Bound States at Threshold Resulting from Coulomb Repulsion

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Abstract

The eigenvalue absorption for a many–particle Hamiltonian depending on a parameter is analyzed in the framework of non–relativistic quantum mechanics. The long–range part of pair potentials is assumed to be pure Coulomb and no restriction on the particle statistics is imposed. It is proved that if the lowest dissociation threshold corresponds to the decay into two likewise non–zero charged clusters then the bound state, which approaches the threshold, does not spread and eventually becomes the bound state at threshold. The obtained results have applications in atomic and nuclear physics. In particular, we prove that an atomic ion with the critical charge Z_{cr} and N_e electrons has a bound state at threshold given that $Z_{cr} \in (N_e - 2, N_e - 1)$, whereby the electrons are treated as fermions and the mass of the nucleus is finite.

I. INTRODUCTION

In Refs. 1 and 2 it was proved that a critically bound N–body system, where none of the subsystems has bound states with $E \leq 0$ and particle pairs have no zero energy resonances, has a square integrable state at zero energy. The condition on the absence of 2–body zero energy resonances was shown to be essential in the three–body case[1]. Here we consider the N–particle system, where particles can be charged and apart from short–range pair–interactions may also interact via Coulomb attraction/repulsion. The formation of bound states at threshold in the two–particle case when the particles Coulomb repel each other is well–studied[3,4]. In the three–particle case there is a well–known proof[5] that a two–electron ion with an infinitely heavy nucleus has a bound state at threshold, when the nuclear charge becomes critical.

Our aim here is to investigate the general many–particle case. Here we generalize the result in Ref. 5 to the case of many electron ions with Fermi statistics and finite nuclear mass. In the proofs we shall use the bounds on Green's functions from Ref. 3 as well as the technique of spreading sequences from Ref. 1, that is we prove the eigenvalue absorption by demonstrating that the wave functions corresponding to bound states do not spread, c.f. Theorem 1 in Ref. 1. A different approach based on the calculus of variations was recently developed in Ref. 6, where the authors give an alternative proof to the result in Ref. 5. The authors in Ref. 6 indicate that their approach could be generalized to the many–particle case. In the present paper as well as in Refs. 5 and 6 one uses essentially the same idea, namely, one uses the fact that the weak limit of ground state wave functions is a solution to the Schrödinger equation at the threshold. The hardest part is to prove that the weak limit is not identically zero. Our approach differs from the ones in Refs. 5 and 6 in that we use the upper bounds on the two–particle Green's functions[3].

The paper is organized as follows. In Sec. II we introduce notations, formulate the main theorem and prove a number of technical lemmas. In Sec. III we derive an upper bound on the Green's function, which is used in Sec. IV for the proof of Theorem 1. In Sec. V we discuss two main applications of Theorem 1 concerning the stability diagram of three Coulomb charges (Theorem 2 in Sec. V A) and negative atomic ions (Theorem 3 in Sec. V B). In Appendix A we derive various criteria for non–spreading sequences.

Let us mention physical applications. The effect when a size of a bound system increases

near the threshold and by far exceeds the scales set by attractive parts of potentials was discovered in neutron halos, helium dimer, Efimov states, for discussion see Refs. 7–10. Here we demonstrate that in a many–particle system similarly to the two–body case[3,4] a Coulomb repulsion between possible decay products blocks the spreading of bound states and forces an L^2 bound state at threshold. In nuclear physics, this, in particular, explains why contrary to neutron halos no proton halos are found[11].

II. FORMULATION OF THE MAIN THEOREM

We consider the N–particle Hamiltonian $(N \geq 3)$

$$H(\lambda) = H_0 + V(\lambda), \tag{1}$$

$$V(\lambda) := \sum_{1 \leq i < j \leq N} V_{ij}(\lambda) \equiv \sum_{1 \leq i < j \leq N} \left[U_{ij}(\lambda; r_i - r_j) + \frac{q_i(\lambda) q_j(\lambda)}{|r_i - r_j|} \right], \tag{2}$$

where $\lambda \in \mathbb{R}$ is a parameter, H_0 is the kinetic energy operator with the center of mass removed, $r_i \in \mathbb{R}^3$ are particles' position vectors and $q_i(\lambda) \in \mathbb{R}$ denote the particles' charges depending on λ. We shall assume that $U_{ij}(\lambda; r) \in L^2(\mathbb{R}^3) + L^\infty_\infty(\mathbb{R}^3)$ for each given λ. Here $L^\infty_\infty(\mathbb{R}^n)$ denotes the space of bounded Borel functions vanishing at infinity. We shall also take particle spins into account, though we shall consider only spin–independent Hamiltonians. The Hamiltonian acts in $L^2(\mathbb{R}^{3N-3}) \oplus L^2(\mathbb{R}^{3N-3}) \oplus \cdots \oplus L^2(\mathbb{R}^{3N-3}) \equiv L^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s})$, where the direct sum has $n_s = (2s_1 + 1)(2s_2 + 1) \ldots (2s_N + 1)$ summands and s_i denotes the spin of particle i. Similar notation for the Hilbert space can be found in Refs. 12 and 13. By Kato's theorem[15,16] $H(\lambda)$ is self–adjoint on $D(H_0) = \mathcal{H}^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s}) \subset L^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s})$, where $\mathcal{H}^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s}) \equiv \mathcal{H}^2(\mathbb{R}^{3N-3}) \oplus \cdots \oplus \mathcal{H}^2(\mathbb{R}^{3N-3})$ and $\mathcal{H}^2(\mathbb{R}^{3N-3})$ denotes the corresponding Sobolev space[16,17]. A function $f \in L^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s})$ depends explicitly on the arguments as $f(x, \sigma_1, \ldots, \sigma_N)$, where $x \in \mathbb{R}^{3N-3}$ and $\sigma_i \in \{\frac{s_i}{2}, \frac{s_i}{2} - 1, \ldots, -\frac{s_i}{2}\}$ are the spin variables.

We treat the particles with integer spins as bosons and particles with half–integer spin as fermions. $\mathcal{P}$ denotes the orthogonal projection operator on the subspace of functions, which are symmetric with respect to the interchange of bosons and antisymmetric with respect to the interchange of fermions. We denote the bottom of the continuous spectrum by

$$E_{thr}(\lambda) := \inf \sigma_{ess} H(\lambda) \mathcal{P}. \tag{3}$$

We shall use the function $\eta_\alpha : \mathbb{R}^n \to \mathbb{R}$, which determines the asymptotic behavior at

infinity

$$\eta_\alpha(r) := \chi_{\{r|\ |r|\leq 1\}} + \chi_{\{r|\ |r|>1\}}|r|^\alpha, \tag{4}$$

where $r \in \mathbb{R}^n, \alpha \in \mathbb{R}_+$ and χ_A always denotes the characteristic function of the set A. Note that $\eta_\alpha(r)$ is continuous and $\eta_{\alpha_1}\eta_{\alpha_2} = \eta_{\alpha_1\alpha_2}$. We make the following assumptions

R1 $H(\lambda)$ is defined for an infinite sequence of parameter values $\lambda_1, \lambda_2, \ldots$ and λ_{cr}, where $\lim_{n\to\infty} \lambda_n = \lambda_{cr}$. For all λ_n there is $E(\lambda_n) \in \mathbb{R}, \psi_n \in D(H_0)$ such that $H(\lambda_n)\psi_n = E(\lambda_n)\psi_n$, where $\|\psi_n\| = 1$, $\mathcal{P}\psi_n = \psi_n$ and $E(\lambda_n) < E_{thr}(\lambda_n)$. Besides, $\lim_{n\to\infty} E(\lambda_n) = \lim_{n\to\infty} E_{thr}(\lambda_n) = E_{thr}(\lambda_{cr})$.

R2 $\sup_{\lambda=\lambda_n,\lambda_{cr}} |U_{ij}(\lambda; y)| \leq \tilde{U}(y)$ and $\sup_{\lambda=\lambda_n,\lambda_{cr}} |q_i(\lambda)q_j(\lambda)| \leq q_0$, where $\tilde{U}(y)$ is such that $\eta_\delta(y)\tilde{U}(y) \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and $\delta \in (3/2, 2)$, $q_0 \in (0, \infty)$ are fixed constants. Additionally, $\lim_{n\to\infty}\big\|\big[V(\lambda_n) - V(\lambda_{cr})\big]f\big\| = 0$ for all $f \in C_0^\infty(\mathbb{R}^{3N-3})$.

Let $a = 1, 2, \ldots, (2^{N-1} - 1)$ label all the distinct ways[14] of partitioning particles into two non–empty clusters $\mathfrak{C}_1^a$ and $\mathfrak{C}_2^a$. We define the Jacobi intercluster coordinates for the clusters $\mathfrak{C}_{1,2}^a$ as $x_i^{a,1}$ and $x_j^{a,2}$ respectively, where $i = 1, 2, \ldots, (\#\mathfrak{C}_1^a - 1)$ and $j = 1, 2, \ldots, (\#\mathfrak{C}_2^a - 1)$ (the symbol $\#$ denotes the number of particles in the corresponding cluster). By x^a we denote the full set of intercluster coordinates and we set

$$|x^a| = \sum_{i=1}^{\#\mathfrak{C}_1^a-1} |x_i^{a,1}| + \sum_{j=1}^{\#\mathfrak{C}_2^a-1} |x_j^{a,2}|. \tag{5}$$

R_a points from the center of mass of $\mathfrak{C}_1^a$ to the center of mass of $\mathfrak{C}_2^a$. The full set of Jacobi coordinates is $(x^a, R_a) \in \mathbb{R}^{3N-3}$.

We denote the sum of interaction cross terms between the clusters by

$$I_a(\lambda) := \sum_{\substack{i\in\mathfrak{C}_1^a \\ j\in\mathfrak{C}_2^a}} V_{ij}(\lambda). \tag{6}$$

The product of net charges of the clusters is defined as

$$Q^a(\lambda) := \sum_{i\in\mathfrak{C}_1^a} \sum_{j\in\mathfrak{C}_2^a} q_i(\lambda)q_j(\lambda). \tag{7}$$

The projection operators on the proper symmetry subspace for the particles within clusters $\mathfrak{C}_1^{(a)}$ and $\mathfrak{C}_2^{(a)}$ are $\mathcal{P}_1^{(a)}$ and $\mathcal{P}_2^{(a)}$ respectively. Namely, $\mathcal{P}_i^{(a)}$ projects on a subspace

of functions, which are antisymmetric with respect to the interchange of fermions in $\mathfrak{C}_i^{(a)}$ and symmetric with respect to the interchange of bosons in $\mathfrak{C}_i^{(a)}$ ($i = 1, 2$). Naturally, $\mathcal{P}\mathcal{P}_{1,2}^{(a)} = \mathcal{P}_{1,2}^{(a)}\mathcal{P} = \mathcal{P}$ and $[\mathcal{P}_1^{(a)}, \mathcal{P}_2^{(a)}] = 0$. We also define $\mathcal{P}^{(a)} := \mathcal{P}_1^{(a)}\mathcal{P}_2^{(a)}$. The Hamiltonian (1) can be decomposed in the following way

$$H(\lambda) = H_{thr}^{(a)}(\lambda) - \frac{\hbar^2}{2\mu_a}\Delta_{R_a} + I_a(\lambda), \tag{8}$$

where $H_{thr}^{(a)}(\lambda)$ is the Hamiltonian of the clusters' intrinsic motion and μ_a denotes the reduced mass derived from clusters' total masses. From now on without loss of generality we set $\hbar^2/(2\mu_a) = 1$.

It is convenient to treat the Hilbert space as the tensor product $L^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s}) = L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s}) \otimes L^2(\mathbb{R}^3)$, where the first term in the product corresponds to the space associated with x^a coordinates and spin variables, while the second one refers to the space associated with the R_a coordinate. In such case the operator $H_{thr}^{(a)}$ has the form $H_{thr}^{(a)} = H_{thr}^a \otimes 1$, where H_{thr}^a is the restriction of $H_{thr}^{(a)}$ to $L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s})$. The coordinate R_a is unaffected by permutations of particles within the clusters $\mathfrak{C}_1^{(a)}$ or $\mathfrak{C}_2^{(a)}$. Therefore, $\mathcal{P}^{(a)} = \mathcal{P}^a \otimes 1$, where $\mathcal{P}^a$ denotes the restriction of $\mathcal{P}^{(a)}$ to the space associated with x^a coordinates and spin variables.

The set of assumptions is continued as follows.

R3 For $\lambda = \lambda_n, \lambda_{cr}$ and $a = 1, \ldots, \mathfrak{N}$ one has $\inf \sigma(H_{thr}^a(\lambda)\mathcal{P}^a) = E_{thr}(\lambda)$. There is $|\Delta\epsilon| > 0$ such that the following inequalities hold for $\lambda = \lambda_n, \lambda_{cr}$

$$\inf \sigma_{ess}(H_{thr}^a(\lambda)\mathcal{P}^a) \geq E_{thr}(\lambda) + 2|\Delta\epsilon| \quad (a = 1, \ldots, \mathfrak{N}), \tag{9}$$

$$\left[H_{thr}^a(\lambda) - E_{thr}(\lambda)\right]\mathcal{P}^a \geq |\Delta\epsilon|\mathcal{P}^a \quad (a = \mathfrak{N} + 1, \ldots, 2^{N-1} - 1). \tag{10}$$

The requirement R3 says that the bottom of the continuous spectrum of $H(\lambda)$ is set by the decomposition into those two clusters that correspond to any of the decompositions $a = 1, \ldots, \mathfrak{N}$. Inequality (9) introduces a gap between the ground state energy of the two clusters and other excited states. For $a = 1, \ldots, \mathfrak{N}$ and $\lambda = \lambda_n, \lambda_{cr}$ we define the projection operator acting on $L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s})$

$$P_{thr}^a(\lambda) = \mathbb{P}_{[E_{thr}(\lambda), E_{thr}(\lambda) + |\Delta\epsilon|]}^a, \tag{11}$$

where $\{\mathbb{P}_\Omega^a\}$ are spectral projections of $H_{thr}^a(\lambda)\mathcal{P}^a$. Note that by R3 the projection operators $P_{thr}^a(\lambda_n)$, $P_{thr}^a(\lambda_{cr})$ have a finite dimensional range.

5

The last assumption introduces the uniform control over the fall off of clusters' wave functions

R4 There are constants $A, \beta > 0$ such that

$$\left\| e^{\beta|x^a|} P_{thr}^a(\lambda) \right\| \leq A \tag{12}$$

for $\lambda = \lambda_n, \lambda_{cr}$ and $a = 1, 2, \ldots, \mathfrak{N}$.

Due to R3 there must exist orthonormal $\varphi_i^a(\lambda) \in D(-\Delta) \subset L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s})$ for $i = 1, 2, \ldots, n_a(\lambda)$ such that

$$P_{thr}^a(\lambda) = \sum_{i=1}^{n_a(\lambda)} E_i^a(\lambda) \varphi_i^a(\lambda) \big(\cdot, \varphi_i^a(\lambda) \big) \quad \text{for } a = 1, \ldots, \mathfrak{N} \text{ and } \lambda = \lambda_n, \lambda_{cr}, \tag{13}$$

where $E_i^a(\lambda) \in [E_{thr}(\lambda), E_{thr}(\lambda) + |\Delta\epsilon|]$. Note that $H_{thr}^a(\lambda_n)\varphi_i^a(\lambda_n) = E_i^a(\lambda_n)\varphi_i^a(\lambda_n)$, therefore, $\| - \Delta\varphi_i^a(\lambda_n)\|$ is uniformly bounded, c. f. Lemma 1 in Ref. 1. Applying Lemma 1 below and using R4 we conclude that there exists an integer ω such that $n_a(\lambda_{cr}), n_a(\lambda_n) \leq \omega$.

Lemma 1. *Suppose that the orthonormal set of function $\phi_1, \ldots, \phi_\mathcal{N} \in D(-\Delta) \subset L^2(\mathbb{R}^d; \mathbb{C}^{n_s})$ is such that $\| - \Delta\phi_i\| \leq T$ and $\|e^{\beta|x|}\phi_i\| \leq A$ for $i = 1, \ldots, \mathcal{N}$, where $T, A, \beta > 0$ are constants. If $d \geq 3$ then*

$$\mathcal{N} \leq C_d \frac{(2T)^{d/2} |\ln 2A|^d}{(2\beta)^d} n_s, \tag{14}$$

where C_d is the Lieb's constant in the Cwikel–Lieb–Rosenbljum bound.

Proof. From $\|e^{\beta|x|}\phi_i\| \leq A$ it follows that

$$(\phi_i, \chi_{\{x| \, |x|\leq R\}}\phi_i) \geq \frac{1}{2}, \tag{15}$$

where we set $R := (\ln 2A)/(2\beta)$. Hence,

$$\left(\phi_i, \left[-\Delta - 2T\chi_{\{x| \, |x|\leq R\}} \right] \phi_i \right) < 0. \tag{16}$$

By the min–max principle $\mathcal{N}$ does not exceed the number of negative energy bound states of the operator in square brackets in (16). This number, in turn, is equal to the number of negative energy bound states of the operator in square brackets considered in $L^2(\mathbb{R}^d)$ times n_s due to the spin degeneracy. Now (14) follows from the Cwikel–Lieb–Rosebljum bound[15,18]. $\qquad\square$

Now we can formulate the main theorem.

Theorem 1. *Suppose that $H(\lambda)$ satisfies $R1 - R4$ and*

$$Q_0 := \inf_{a=1,\dots,\mathfrak{N}} \inf_{\lambda=\lambda_n,\lambda_{cr}} Q^a(\lambda) > 0. \tag{17}$$

Then the sequence ψ_n does not spread and there exists $\psi_{cr} \in D(H_0) \subset L^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s})$ such that $H(\lambda_{cr})\psi_{cr} = E_{thr}(\lambda_{cr})\psi_{cr}$, where $\|\psi_{cr}\| = 1$ and $\psi_{cr} = \mathcal{P}\psi_{cr}$.

Let us remark that the term spreading was defined in Ref. 1 for sequences in $L^2(\mathbb{R}^d)$. We shall say that a sequence $f_n \in L^2(\mathbb{R}^d; \mathbb{C}^{n_s})$ spreads if $f_n(x, \sigma_1, \dots, \sigma_N)$ spreads for all possible fixed values of the spin variables. We postpone the proof of Theorem 1 to Sec. IV.

Together with the upper bound on the Green's function derived in the next section the following lemma is the key ingredient in the proof of Theorem 1.

Lemma 2. *There is $\Theta_a(x) \in L^2(\mathbb{R}^{3N-3}) + L^\infty_\infty(\mathbb{R}^{3N-3})$ independent of λ such that*

$$\left| e^{-\beta|x^a|} \eta_\delta(R_a) \left[I_a(\lambda) - Q^a(\lambda)\eta_{-1}(R_a) \right] \right| \le \Theta_a(x) \tag{18}$$

for $\lambda = \lambda_n, \lambda_{cr}$ defined in R1, δ defined in R2 and β defined in R4.

Proof. The statement of the lemma is based on the following inequality, which can be checked directly. For all $s, s' \in \mathbb{R}^3$

$$\left| \frac{\chi_{\{s,s' \mid |s-s'| \ge 1\}}}{|s - s'|} - \eta_{-1}(s) \right| \le 2\eta_2(s')\eta_{-2}(s). \tag{19}$$

For fixed s' the term on the lhs of (19) falls off like $|s|^{-2}$. We write

$$\left| I_a(\lambda) - Q^a(\lambda)\eta_{-1}(R_a) \right| \le \sum_{\substack{i\in\mathfrak{C}_1^a \\ j\in\mathfrak{C}_2^a}} \tilde{U}_{ij} + \sum_{\substack{i\in\mathfrak{C}_1^a \\ j\in\mathfrak{C}_2^a}} \frac{q_0}{|r_i - r_j|} \chi_{\{x\mid |r_i-r_j|\le 1\}}$$

$$+ \sum_{\substack{i\in\mathfrak{C}_1^a \\ j\in\mathfrak{C}_2^a}} |q_i(\lambda)q_j(\lambda)| \left| \frac{\chi_{\{x\mid |r_i-r_j|\ge 1\}}}{|r_i - r_j|} - \eta_{-1}(R_a) \right|. \tag{20}$$

For any cluster decomposition a and $i \in \mathfrak{C}_1^a, j \in \mathfrak{C}_2^a$

$$r_j - r_i = R_a + \sum_{i=1}^{\#\mathfrak{C}_1^a - 1} c_i^{a,1} x_i^{a,1} + \sum_{i=1}^{\#\mathfrak{C}_2^a - 1} c_i^{a,2} x_i^{a,2}, \tag{21}$$

7

where $c_i^{a,1}$, $c_i^{a,2}$ are numerical coefficients depending on masses. It is easy to see that the coefficient in front of R_a is always 1 by fixing $|x^a|$ and taking $|R_a| \gg 1$. Therefore, by (19) we have

$$\left| \frac{\chi_{\{x|\ |r_i-r_j|\geq 1\}}}{|r_i - r_j|} - \eta_{-1}(R_a) \right| \leq c_0 \eta_2(|x^a|)\eta_{-2}(R_a), \tag{22}$$

where $c_0 > 0$ is some constant. Substituting (22) into (20) we conclude that the inequality (18) would be true if we set $\Theta_a = \Theta_{a1} + \Theta_{a2}$, where

$$\Theta_{a1}(x) := e^{-\beta|x^a|}\eta_\delta(R_a) \sum_{\substack{i\in\mathfrak{C}_1^a \\ j\in\mathfrak{C}_2^a}} \left[\tilde{U}_{ij} + \frac{q_0}{|r_i - r_j|}\chi_{\{x|\ |r_i-r_j|\leq 1\}} \right], \tag{23}$$

$$\Theta_{a2}(x) := c_0 N(N-1)q_0 e^{-\beta|x^a|}\eta_2(|x^a|)\eta_{\delta-2}(R_a). \tag{24}$$

Using R2 it is easy to see that $\Theta_{a1} \in L^2(\mathbb{R}^{3N-3}) + L^\infty_\infty(\mathbb{R}^{3N-3})$. Because $\delta < 2$ we have $\Theta_{a2} \in L^\infty_\infty(\mathbb{R}^{3N-3})$. $\qquad\square$

III.   UPPER BOUND ON THE TWO PARTICLE GREEN'S FUNCTION

Consider the following integral operator on $L^2(\mathbb{R}^3)$

$$G_k^c(A) = \left[-\Delta + A\eta_{-1}(r) + k^2 \right]^{-1}, \tag{25}$$

for $A, k > 0$, whose integral kernel we denote as $G_k^c(A; r, r')$ (the superscript "c" refers to "Coulomb"). Note that $G_k^c(A; r, r') \leq G_{\tilde{k}}^c(\tilde{A}; r, r')$ away from $r = r'$ if either $\tilde{A} \leq A$ or $\tilde{k} \leq k$, c. f. Lemma 1 in Ref. 3. The following Lemma uses the upper bound on a two particle Green's function from Ref. 3.

Lemma 3. *There is $b(A) > 0$ such that for all $A > 0$, $n > 0$*

$$\sup_{k>0} \left\| G_k^c(A)\chi_{\{r|\ |r|\leq n\}} \right\| \leq b(A)n, \tag{26}$$

where the norm on the lhs is the operator norm.

Proof. The operator $G_k^c(A)$ is an integral operator with a positive kernel[19] and, hence, it suffices to consider (26) for $n > 1$. For a shorter notation we denote $\chi_n := \chi_{\{r|\ |r|\leq n\}}$. Obviously

$$\|G_k^c(A)\chi_n\| \leq \|\chi_{4n} G_k^c(A)\chi_n\| + \|(1 - \chi_{4n})G_k^c(A)\chi_n\|$$

$$\leq \|\chi_{4n} G_k^c(A)\chi_{4n}\| + \|(1 - \chi_{4n})G_k^c(A)\chi_n\|, \tag{27}$$

where the last inequality follows from $G_k^c(A)$ being an integral operator with a positive kernel[3,19]. We shall derive the following estimates $\|\chi_{4n} G_k^c(A)\chi_{4n}\| = \mathcal{O}(n)$ and $\|(1 - \chi_{4n})G_k^c(A)\chi_n\| = \mathrm{o}(n)$ for $n \to \infty$, from which the the statement of the Lemma follows. The first term on the rhs of (27) is the norm of the self–adjoint operator, which can be estimated as follows

$$\|\chi_{4n} G_k^c(A)\chi_{4n}\| = \sup_{\|f\|=1} \left(\chi_{4n}f, G_k^c(A)\chi_{4n}f\right) \leq$$

$$\sup_{\|f\|=1} \left(\chi_{4n}f, (A\eta_{-1})^{-1}\chi_{4n}f\right) = 4A^{-1}n, \tag{28}$$

where we have used the inequality $(B + \varepsilon)^{-1} \leq (C + \varepsilon)^{-1}$ for non–negative self–adjoint operators $B \geq C \geq 0$ and $\varepsilon > 0$ (see, for example Ref. 20, Proposition A.2.5 on page 131). Thus $\|\chi_{4n} G_k^c(A)\chi_{4n}\| = \mathcal{O}(n)$ as claimed.

Let us now consider the second term on the rhs of (27). We shall need the bound on the Green's function from Ref. 3. Let $\tilde{G}_k(a; r, r')$ denote the integral kernel of the following operator on $L^2(\mathbb{R}^3)$

$$\tilde{G}_k(a) = \left[-\Delta + \left(\frac{a^2}{4}|r|^{-1} + \frac{a}{4}|r|^{-3/2}\right)\chi_{\{r|\,|r|\geq 1\}} + k^2\right]^{-1}. \tag{29}$$

Lets us set a equal to the positive root of the equation $a(a + 1) = 4A$. Then we get

$$A\eta_{-1}(r) \geq \left(\frac{a^2}{4}|r|^{-1} + \frac{a}{4}|r|^{-3/2}\right)\chi_{\{r|\,|r|\geq 1\}}, \tag{30}$$

which means that $G_k^c(A; r, r') \leq \tilde{G}_k(a; r, r')$ pointwise for all $r \neq r'$, see Ref. 3. The upper bound on $\tilde{G}_k(a; r, r')$ from Ref. 3 (Eqs.(42)–(43) and Eqs. (39)–(40) in Ref. 3) reads

$$\tilde{G}_k(a; r, r') \leq \frac{1}{4\pi|r - r'|} \times \begin{cases} 1 & \text{for } |r - r'| \leq \tilde{R}_0 \\ \exp\left\{\tilde{a}\sqrt{\tilde{R}_0} - \tilde{a}\sqrt{|r - r'|}\right\} & \text{for } |r - r'| > \tilde{R}_0, \end{cases} \tag{31}$$

where $\tilde{R}_0, \tilde{a}$ have to be chosen to satisfy the following inequalities

$$\tilde{R}_0 \geq 1 + |r'|, \tag{32}$$

$$\tilde{a} \leq a\tilde{R}_0^{3/2}(\tilde{R}_0 + |r'|)^{-3/2}. \tag{33}$$

From the inequality (31) we obtain the bound

$$\tilde{G}_k(a; r, r')\chi_{\{|r'|\leq n\}} \leq \frac{1}{4\pi|r - r'|} \times \begin{cases} 1 & \text{for } |r - r'| \leq 2n \\ \exp\left\{\frac{a}{2}\left(\sqrt{2n} - \sqrt{|r - r'|}\right)\right\} & \text{for } |r - r'| > 2n, \end{cases} \tag{34}$$

9

where we have set $\tilde{R}_0 = 2n$ and $\tilde{a} = a/2$. It is straightforward to check that this choice of $\tilde{R}_0, \tilde{a}$ indeed satisfies (32)–(33). Taking into account that $G_k^c(A; r, r') \leq \tilde{G}_k(a; r, r')$ we finally get from (34) the required bound

$$G_k^c(A; r, r') \chi_{\{r, r' \mid |r| \geq 4n, |r'| \leq n\}} \leq \frac{e^{\frac{a}{2}\left(\sqrt{2n} - \sqrt{|r| - n}\right)}}{4\pi(3n)} \chi_{\{r, r' \mid |r| \geq 4n, |r'| \leq n\}}. \tag{35}$$

Note that the rhs of (35) does not depend on k. Using the upper bound (35) and estimating the operator norm through the Hilbert–Schmidt norm we get

$$\begin{aligned}
\|(1 - \chi_{4n}) G_k^c(A) \chi_n\|^2 &\leq \int_{|r| \geq 4n} dr \int_{|r'| \leq n} dr' |G_k^c(A; r, r')|^2 \\
&\leq \frac{n}{27} e^{a\sqrt{2n}} \int_{4n}^{\infty} e^{-a\sqrt{t-n}} t^2 dt.
\end{aligned} \tag{36}$$

The integral in (36) can be calculated explicitly and we obtain $\|(1 - \chi_{4n}) G_k^c(A) \chi_n\| = o(n)$ as claimed. $\qquad\square$

We shall need the following corollary of Lemma 3

Lemma 4. *For fixed $A > 0, \alpha > 3/2$ the following inequality holds*

$$\sup_{k > 0} \left\| G_k^c(A) \, \eta_{-\alpha} \right\| < \infty. \tag{37}$$

Proof. For an arbitrary $f \in L^2(\mathbb{R}^3)$ we have

$$\begin{aligned}
\left\| G_k^c(A) \eta_{-\alpha} f \right\| &= \lim_{N \to \infty} \left\| \sum_{n=1}^{N} G_k^c(A) \eta_{-\alpha} (\chi_n - \chi_{n-1}) f \right\| \\
&\leq \lim_{N \to \infty} \sum_{n=1}^{N} \left\| G_k^c(A) \chi_n \eta_{-\alpha} (\chi_n - \chi_{n-1})^2 f \right\|,
\end{aligned} \tag{38}$$

where we have used $(\chi_n - \chi_{n-1})^2 = (\chi_n - \chi_{n-1})$ and $\chi_n(\chi_n - \chi_{n-1}) = (\chi_n - \chi_{n-1})$. For the operator norms we have $\|\eta_{-\alpha}\chi_1\| = 1$ and $\|\eta_{-\alpha}(\chi_n - \chi_{n-1})\| = (n-1)^{-\alpha}$ for $n \geq 2$. Substituting these into (38) and using Lemma 3 we rewrite (38) as

$$\left\| G_k^c(A) \eta_{-\alpha} f \right\| \leq b(A) \lim_{N \to \infty} \left(\|\chi_1 f\| + \sum_{n=2}^{N} n(n-1)^{-\alpha} \|(\chi_n - \chi_{n-1}) f\| \right). \tag{39}$$

Now using that $\sum_n \|(\chi_n - \chi_{n-1}) f\|^2 = \|f\|^2$ and applying the Cauchy-Schwartz inequality we get from Eq. (39)

$$\left\| G_k^c(A) \eta_{-\alpha} \right\| \leq b(A) \left(1 + \sum_{n=2}^{\infty} n^2 (n-1)^{-2\alpha} \right)^{1/2}. \tag{40}$$

For $\alpha > 3/2$ the series on the rhs of Eq. (40) obviously converge. $\qquad\square$

IV. PROOF OF THE MAIN THEOREM

We shall need an analogue of the IMS localization formula, see Ref. 14. The functions $J_a \in C^\infty(\mathbb{R}^{3N-3})$ form the partition of unity $\sum_a J_a^2 = 1$ and are homogeneous of degree zero in the exterior of the unit sphere, *i.e.* $J_a(\lambda x) = J_a(x)$ for $\lambda \geq 1$, $|x| = 1$ (this makes $|\nabla J_a|$ fall off at infinity). Additionally, there exists a constant $C > 0$ such that

$$\text{supp} J_a \cap \{x | |x| > 1\} \subset \{x | |x_i - x_j| \geq C|x| \quad \text{for } i \in \mathfrak{C}_1^a, j \in \mathfrak{C}_2^a\}. \tag{41}$$

The functions of the IMS decomposition can be chosen[21,22] invariant under permutations of particle coordinates both in $\mathfrak{C}_1^a$ and in $\mathfrak{C}_2^a$, hence $[J_a, \mathcal{P}^{(a)}] = 0$. Note also that for all $f \in \mathcal{H}^2(\mathbb{R}^{3N-3})$ one has $J_a f \in \mathcal{H}^2(\mathbb{R}^{3N-3})$, c. f. Lemma 7.4 in Ref. 17 (the proof in Ref. 17 easily extends to the case of Sobolev spaces of higher order). The following version of the IMS localization formula can be verified by the direct substitution

$$\Delta = \Delta \sum_{a,b} J_a^2 J_b^2 = \sum_{a,b} J_a J_b \Delta J_b J_a + 2 \sum_a |\nabla J_a|^2, \tag{42}$$

where Δ is the Laplace on $\mathbb{R}^{3N-3}$. Rescaling (42) we get

$$H_0 = \sum_{a,b} J_a J_b H_0 J_b J_a + 2 \sum_a \sum_{s=1}^{3N-3} \mathfrak{m}_s |\partial_s J_a|^2, \tag{43}$$

where $\mathfrak{m}_s$ are real coefficients depending on masses and the second term is relatively H_0 compact[14]. We introduce

$$H_a(\lambda) = H(\lambda) - I_a(\lambda) = H_{thr}^{(a)}(\lambda) - \Delta_{R_a}. \tag{44}$$

From (43) it follows that

$$H(\lambda) = \sum_a J_a^2 H_a(\lambda) J_a^2 + \sum_{a \neq b} J_a J_b H_{ab}(\lambda) J_b J_a + K(\lambda), \tag{45}$$

where we define

$$K(\lambda) := \sum_{a \neq b} J_a^2 J_b^2 I_{ab}(\lambda) + \sum_a \left[J_a^4 I_a(\lambda) + 2 \sum_{s=1}^{3N-3} \mathfrak{m}_s |\partial_s J_a|^2 \right], \tag{46}$$

$$H_{ab}(\lambda) := H_0 + \sum_{s=1}^2 \sum_{p=1}^2 \sum_{i,j \in \mathfrak{C}_s^a \cap \mathfrak{C}_p^b} V_{ij}(\lambda), \tag{47}$$

$$I_{ab}(\lambda) := H(\lambda) - H_{ab}(\lambda). \tag{48}$$

11

The Hamiltonian H_{ab} defined for $a \neq b$ contains intercluster interactions of the following four clusters $\mathfrak{C}_1^a \cap \mathfrak{C}_1^b$, $\mathfrak{C}_1^a \cap \mathfrak{C}_2^b$, $\mathfrak{C}_2^a \cap \mathfrak{C}_1^b$, $\mathfrak{C}_2^a \cap \mathfrak{C}_2^b$, while all interaction cross–terms between these four clusters are contained in I_{ab}. (For some partitions it might happen that one of the four clusters is empty). If we define by $\mathcal{P}_{sp}^{(ab)}$ the projection operator on the proper symmetry subspace for particles within the cluster $\mathfrak{C}_s^a \cap \mathfrak{C}_p^b$ then by the HVZ theorem[15,16,23]

$$\inf \sigma \left(H_{ab}(\lambda) \mathcal{P}^{(ab)} \right) \geq E_{thr}(\lambda), \tag{49}$$

where we define

$$\mathcal{P}^{(ab)} := \mathcal{P}_{11}^{(ab)} \mathcal{P}_{12}^{(ab)} \mathcal{P}_{21}^{(ab)} \mathcal{P}_{22}^{(ab)}. \tag{50}$$

Note that $[J_a J_b, \mathcal{P}^{(ab)}] = 0$.

Lemma 5. *Suppose that $H(\lambda)$ satisfies $R1 - R4$. If $\psi_n \xrightarrow{w} \phi_0$ then*

$$\lim_{n \to \infty} \left\| \left[1 - P_{thr}^{(a)}(\lambda_n) \right] J_a^2 (\psi_n - \phi_0) \right\| = 0 \quad (a = 1, \ldots, \mathfrak{N}), \tag{51}$$

$$\lim_{n \to \infty} \left\| J_a^2 (\psi_n - \phi_0) \right\| = 0 \quad (a = \mathfrak{N} + 1, \ldots, 2^{N-1} - 1). \tag{52}$$

Proof. Note that $\phi_0 \in D(H_0)$ by Lemmas 1, 2(a) in Ref. 1. For every $g \in L^2(\mathbb{R}^{3N-3}; \mathbb{C}^{n_s})$ we have

$$\left(g, [1 - \mathcal{P}] \phi_0 \right) = \left([1 - \mathcal{P}] g, \phi_0 \right)$$

$$= \lim_{n \to \infty} \left([1 - \mathcal{P}] g, \psi_n \right) = \lim_{n \to \infty} \left(g, [1 - \mathcal{P}] \psi_n \right) = 0. \tag{53}$$

Because g in (53) is arbitrary we conclude that $\mathcal{P} \phi_0 = \phi_0$. Consequently, $\mathcal{P}^{(a)} \phi_0 = \phi_0$.

Following the arguments of the proof of Lemma 10 in Ref. 1 we get

$$\lim_{n \to \infty} \left((\psi_n - \phi_0), [H(\lambda_n) - E_{thr}(\lambda_n)] (\psi_n - \phi_0) \right) = 0. \tag{54}$$

We define $\tilde{K}$ exactly as $K(\lambda)$ in (46), except that all V_{ij} entering K via $I_a(\lambda)$ and $I_{ab}(\lambda)$ are replaced with

$$\tilde{V}_{ij} := \tilde{U}_{ij} + \frac{q_0}{|x_i - x_j|}, \tag{55}$$

where $\tilde{U}_{ij} := \tilde{U}(x_i - x_j)$. Then $\tilde{K}$ does not depend on λ and is relatively H_0 compact. Besides for all $f \in D(H_0)$ one has $|(f, K(\lambda)f)| \leq (f, \tilde{K}f)$. Thus

$$\lim_{n \to \infty} \left((\psi_n - \phi_0), K(\lambda_n)(\psi_n - \phi_0) \right) = 0, \tag{56}$$

12

see the proof of Lemma 10 in Ref. 1. Substituting (45) into (54) and using (56) yields

$$\lim_{n \to \infty} \sum_a \left((\psi_n - \phi_0), J_a^2 \left[H_a(\lambda_n) - E_{thr}(\lambda_n) \right] J_a^2 (\psi_n - \phi_0) \right)$$

$$+ \sum_{a \neq b} \left((\psi_n - \phi_0), J_a J_b \left[H_{ab}(\lambda_n) - E_{thr}(\lambda_n) \right] J_b J_a (\psi_n - \phi_0) \right) = 0. \tag{57}$$

The scalar products under the first sum are clearly non–negative. The terms under the second sum are non–negative by (49) (one can insert $\mathcal{P}^{(ab)}$ because $\mathcal{P}^{(ab)} \mathcal{P} = \mathcal{P}$ and $[J_a J_b, \mathcal{P}^{(ab)}] = 0$). Thus we obtain

$$\lim_{n \to \infty} \left((\psi_n - \phi_0), J_a^2 \left[H_a(\lambda_n) - E_{thr}(\lambda_n) \right] J_a^2 (\psi_n - \phi_0) \right) = 0 \tag{58}$$

for all partitions $a = 1, \ldots, 2^{N-1} - 1$. Using (44) and $-\Delta_{R_a}$ being non–negative gives

$$\lim_{n \to \infty} \left((\psi_n - \phi_0), J_a^2 \left[H_{thr}^{(a)}(\lambda_n) - E_{thr}(\lambda_n) \right] \mathcal{P}^{(a)} J_a^2 (\psi_n - \phi_0) \right) = 0, \tag{59}$$

where we have inserted $\mathcal{P}^{(a)}$. For $a \geq \mathfrak{N} + 1$ the statement of the lemma given by (52) easily follows from (59) and R3. To prove (51) it suffices to insert into (59) the identity $1 = P_{thr}^{(a)} + (1 - P_{thr}^{(a)})$ and to use the inequality

$$\left[H_{thr}^{(a)}(\lambda_n) - E_{thr}(\lambda_n) \right] \left(1 - P_{thr}^{(a)}(\lambda_n) \right) \mathcal{P}^{(a)} \geq |\Delta\epsilon| \left(1 - P_{thr}^{(a)}(\lambda_n) \right) \mathcal{P}^{(a)} \quad (a = 1, \ldots, \mathfrak{N}), \tag{60}$$

which follows from (11). $\qquad \square$

Using Lemma 2 we prove

Lemma 6. *Suppose that $H(\lambda)$ satisfies $R1 - R4$ and ψ_n defined in $R1$ converges weakly. Then for $a = 1, 2, \ldots, \mathfrak{N}$ the sequence $P_{thr}^{(a)}(\lambda_n)\psi_n$ does not spread.*

Proof. So let us assume that $\psi_n \overset{w}{\to} \phi_0$, where $\phi_0 \in D(H_0)$, see Lemma 1 in Ref. 1. The Schrödinger equation can be written as

$$\left\{ \left[H_{thr}^{(a)}(\lambda_n) - E_{thr}(\lambda_n) \right] - \Delta_{R_a} + Q^a(\lambda_n)\eta_{-1}(R_a) + k_n^2 \right\} \psi_n$$

$$= - \left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(R_a) \right] \psi_n, \tag{61}$$

where $k_n^2 := E_{thr}(\lambda_n) - E(\lambda_n)$. By Lemma 1 we can write

$$P_{thr}^{(a)}(\lambda_n) = \sum_{i=1}^{\omega} E_i^a(\lambda_n) P_{\varphi_i^a}(\lambda_n), \tag{62}$$

13

where $P_{\varphi_i^a}(\lambda_n) = \varphi_i^a(\cdot, \varphi_i^a) \otimes 1$ and $\varphi_i^a(\lambda_n)$ are orthonormal eigenstates of $H_{thr}^{(a)}(\lambda_n)$ corresponding to eigenvalues $E_i^a(\lambda_n)$ ($\varphi_i^a(\lambda_n) = 0$, where necessary). By (11) we have $E_i^a(\lambda_n) \in [E_{thr}(\lambda_n), E_{thr}(\lambda_n) + |\Delta\epsilon|]$. Because the sum in (62) runs over a finite number of terms (see Lemma 1) to prove the theorem it suffices to show that $P_{\varphi_i^a}(\lambda_n)\psi_n$ does not spread. Acting with $P_{\varphi_i^a}(\lambda_n)$ on both sides of (61) results in

$$\left[-\Delta_{R_a} + Q^a(\lambda_n)\eta_{-1}(R_a) + {k'_n}^2\right] P_{\varphi_i^a}(\lambda_n)\psi_n \tag{63}$$

$$= -P_{\varphi_i^a}(\lambda_n)\left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(r)\right]\psi_n, \tag{64}$$

where $k'_n := \left[|E_i^a(\lambda_n) - E_{thr}(\lambda_n)| + k_n^2\right]^{1/2}$. Acting with

$$G_{k'_n}^c(Q^a(\lambda_n)) := 1 \otimes \left[-\Delta_{R_a} + Q^a(\lambda_n)\eta_{-1}(R_a) + {k'_n}^2\right]^{-1} \tag{65}$$

on both sides of (63)–(64) we get

$$P_{\varphi_i^a}(\lambda_n)\psi_n = -G_{k'_n}^c(Q^a(\lambda_n))\eta_{-\delta}(R_a)P_{\varphi_i^a}(\lambda_n)\eta_\delta(R_a)\left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(R_a)\right]\psi_n, \tag{66}$$

where we have inserted $\eta_\delta\eta_{-\delta} = 1$ (δ is defined in $R2$). Adding and subtracting ϕ_0 from ψ_n we rewrite (66) as inequality

$$\left|P_{\varphi_i^a}(\lambda_n)\psi_n\right|$$
$$\leq \left|G_{k'_n}^c(Q^a(\lambda_n))\eta_{-\delta}(R_a)P_{\varphi_i^a}(\lambda_n)\eta_\delta(R_a)\left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(R_a)\right](\psi_n - \phi_0)\right|$$
$$+ \left|G_{k'_n}^c(Q^a(\lambda_n))\eta_{-\delta}(R_a)P_{\varphi_i^a}(\lambda_n)\eta_\delta(R_a)\left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(R_a)\right]\phi_0\right|. \tag{67}$$

This can be continued as

$$\left|P_{\varphi_i^a}(\lambda_n)\psi_n\right|$$
$$\leq G_{k_n}^c(Q_0)\eta_{-\delta}(R_a)P_{|\varphi_i^a|}(\lambda_n)\left|\eta_\delta(R_a)\left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(R_a)\right](\psi_n - \phi_0)\right|$$
$$+ G_{k_n}^c(Q_0)\eta_{-\delta}(R_a)P_{|\varphi_i^a|}(\lambda_n)\left|\eta_\delta(R_a)\left[I_a(\lambda_n) - Q^a(\lambda_n)\eta_{-1}(R_a)\right]\phi_0\right|, \tag{68}$$

where we define $P_{|\varphi_i^a|} := |\varphi_i^a|(\cdot, |\varphi_i^a|) \otimes 1$ and use $Q^a(\lambda_n) > Q_0$, ${k'_n}^2 \geq k_n^2$ (see remark after Eq. (25)). Finally, applying Lemma 2 we write

$$\left|P_{\varphi_i^a}(\lambda_n)\psi_n\right| \leq g_n + P_{|\varphi_i^a|}(\lambda_n)e^{\beta|x^a|}h_n, \tag{69}$$

where we define

$$g_n := G_{k_n}^c(Q_0)\eta_{-\delta}(R_a)P_{|\varphi_i^a|}(\lambda_n)e^{\beta|x^a|}\Theta_a(x)\left|\psi_n - \phi_0\right|, \tag{70}$$

$$h_n := G_{k_n}^c(Q_0)\eta_{-\delta}(R_a)\Theta_a(x)\left|\phi_0\right|. \tag{71}$$

14

It remains to prove that both terms on the rhs of (69) do not spread. We have

$$\|g_n\| \leq \left\|G^c_{k_n}(Q_0)\eta_{-\delta}(R_a)\right\| \times \left\|P_{|\varphi^a_i|}(\lambda_n)e^{\beta|x^a|}\right\| \times \left\|\Theta_a(x)(\psi_n - \phi_0)\right\|. \tag{72}$$

The first two operator norms are uniformly bounded by Lemma 4 and R4. Note that $\Theta_a(x)$ is relatively H_0 compact by Lemma 2, see Lemma 7.11 in Ref. 16. Therefore, the last norm goes to zero by Lemma 2 in Ref. 1. Hence, $\|g_n\| \to 0$ and g_n does not spread. By the same reasoning the sequence $|h_n|$ is uniformly norm–bounded. The kernel $G^c_{k_n}$ is pointwise dominated by the kernel of $G^c_{k_s}$ if $k_s \leq k_n$, see remark after Eq. (25). Using this fact it is easy to see that h_n satisfies the conditions of Lemma 9 and therefore does not spread. Now the rhs of (69) does not spread by Lemma 10 since $\|e^{\beta|x^a|}P_{|\varphi^a_i|}(\lambda_n)e^{\beta|x^a|}\| \leq A^2$ by R4. $\qquad\square$

Now we can prove the main theorem.

Proof of Theorem 1. We shall first prove that ψ_n does not spread. To prove this, by Lemma 4 of Ref. 1 it is sufficient to show that every weakly converging subsequence of ψ_n converges also in norm. So let us assume that $\psi_{n_k} \overset{w}{\to} \phi_0$, where ψ_{n_k} is some weakly convergent subsequence of ψ_n, $\phi_0 \in D(H_0)$ by Lemmas 1, 2(a) in Ref. 1. (Let us remark that one weakly converging subsequence always exists due to the Banach–Alaoglu theorem). Repeating the arguments in the proof of Lemma 5 (around Eq. 53) we conclude that $\mathcal{P}\phi_0 = \phi_0$. Our next aim is to show that ψ_{n_k} does not spread; by Lemmas 1, 3(a) of Ref. 1 this will imply that $\psi_{n_k} \to \phi_0$ in norm. The following identity is obvious

$$\psi_{n_k} = \sum_{a=1}^{\mathfrak{N}} P^{(a)}_{thr}(\lambda_{n_k})J^2_a(\psi_{n_k} - \phi_0) + \sum_{a=1}^{\mathfrak{N}}\left[1 - P^{(a)}_{thr}(\lambda_{n_k})\right]J^2_a(\psi_{n_k} - \phi_0)$$
$$+ \sum_{a=\mathfrak{N}+1}^{2^{N-1}-1} J^2_a(\psi_{n_k} - \phi_0) + \phi_0. \tag{73}$$

The last two sums on the rhs go to zero in norm by Lemma 5. It suffices to prove that each term in the first sum does not spread. Using $\sum_a J^2_a = 1$ we write

$$\left|P^{(a)}_{thr}(\lambda_{n_k})J^2_a(\psi_{n_k} - \phi_0)\right| \leq \left|P^{(a)}_{thr}(\lambda_{n_k})(\psi_{n_k} - \phi_0)\right| + \sum_{b\neq a}\left|P^{(a)}_{thr}(\lambda_{n_k})J^2_b(\psi_{n_k} - \phi_0)\right|$$
$$\leq \left|P^{(a)}_{thr}(\lambda_{n_k})\psi_{n_k}\right| + \left|P^{(a)}_{thr}(\lambda_{n_k})\phi_0\right| + \sum_{b\neq a}\left|P^{(a)}_{thr}(\lambda_{n_k})J^2_b(\psi_{n_k} - \phi_0)\right|. \tag{74}$$

The first two terms on the rhs of (74) do not spread by Lemmas 6,10 respectively. It remains to show that each term under the sum on the rhs of (74) goes to zero in norm. Indeed, for

$b \neq a$

$$\left\| P_{thr}^{(a)}(\lambda_{n_k}) J_b^2(\psi_{n_k} - \phi_0) \right\| \leq \left\| P_{thr}^{(a)}(\lambda_{n_k}) e^{\beta|x^a|} \right\| \times \left\| e^{-\beta|x^a|} J_b^2(\psi_{n_k} - \phi_0) \right\|. \qquad (75)$$

The operator norm on the rhs is uniformly bounded by $R4$. The second norm goes to zero because $e^{-\beta|x^a|} J_b^2 \in L_\infty^\infty(\mathbb{R}^{3N-3})$ and is thus relatively H_0 compact (c. f. Lemma 2 in Ref. 1 and Lemma 7.11 in Ref. 16). Thus we have proved that $\psi_{n_k} \to \phi_0$ in norm and ψ_n does not spread. By Theorem 1 in Ref. 1 there exists $\psi_{cr} \in D(H_0)$ such that $H(\lambda_{cr})\psi_{cr} = E_{thr}(\lambda_{cr})\psi_{cr}$. From Eqs. (10)–(11) in Ref. 1 it is easy to see that we can set $\psi_{cr} = \phi_0$, which results in $\|\psi_{cr}\| = 1$ and $\psi_{cr} = \mathcal{P}\psi_{cr}$. $\qquad \square$

V.   APPLICATIONS

A.   Three Coulomb charges with finite masses

We consider the Coulomb Hamiltonian of three particles with charges $\{q_1, q_2, -1\}$ and masses $\{m_1, m_2, m_3\}$. We use Jacobi coordinates $\xi = r_3 - r_2$, $R = r_1 - r_2 - s\xi$, where $s = m_3/(m_3 + m_2)$. The Hamiltonian reads[24]

$$H(q_1, q_2) = -\frac{1}{2\mu_{23}}\Delta_\xi - \frac{1}{2\mu}\Delta_R - \frac{q_2}{|\xi|} - \frac{q_1}{|(1-s)\xi - R|} + \frac{q_1 q_2}{|a\xi + R|}, \qquad (76)$$

where $\mu_{ik} = m_i m_k/(m_i + m_k)$, $\mu = m_1(m_2 + m_3)/(m_1 + m_2 + m_3)$ are reduced masses. We keep the masses fixed making $H(q_1, q_2)$ depend on $q_{1,2} \geq 0$. By the Kato's theorem[15,16] $H(q_1, q_2)$ is a self–adjoint operator acting in $L^2(\mathbb{R}^6)$ with the domain $D(H) = \mathcal{H}^2(\mathbb{R}^6)$. The particle spins can be neglected here and in order to apply the previous formalism we simply set all particle spins to zero.

The Hamiltonian $H(q_1, q_2)$ is called *stable* if $\inf \sigma H(q_1, q_2) < E_{thr}(q_1, q_2)$, where $E_{thr}(q_1, q_2) := \inf \sigma_{ess} H(q_1, q_2)$. A typical stability diagram[26] for $H(q_1, q_2)$ is sketched in Fig. 1. The properties of the stability diagram are discussed in detail in Ref. 26. We mention some key features of the stability diagram, for details see Ref. 26. In the square $\{q_{1,2}|\, 0 < q_{1,2} < 1\}$ the Hamiltonian $H(q_1, q_2)$ is always stable (due to long–range attraction between the bound pair and the third particle). The line of equal energy thresholds is determined through $\mu_{23}q_2^2 = \mu_{13}q_1^2$ and divides the plane into upper and lower sectors, where the lowest dissociation threshold corresponds to $\{123\} \to \{23\} + 1$ and $\{123\} \to \{13\} + 2$ respectively. In each sector the stability area is shaped by two arcs, which form a cusp on the line

16

of equal energy thresholds, just like in Fig. 1. The arc in the upper sector starts at $(\mathfrak{q}_1, 1)$ and in the lower sector at $(1, \mathfrak{q}_2)$ and both end up on the line of equal thresholds. The points $\{q_{1,2}|\, 0 < q_1 \leq \mathfrak{q}_1, q_2 = 1\}$ and $\{q_{1,2}|\, 0 < q_2 \leq \mathfrak{q}_2, q_1 = 1\}$ correspond to unstable $H(q_1, q_2)$ and the points $\{q_{1,2}|\, q_1 \geq \mathfrak{q}_1, q_2 = 1\} \cap \{q_{1,2}|\, \mu_{23}q_2^2 \geq \mu_{13}q_1^2\}$ and $\{q_{1,2}|\, q_2 > \mathfrak{q}_2, q_1 = 1\} \cap \{q_{1,2}|\, \mu_{23}q_2^2 \leq \mu_{13}q_1^2\}$ correspond[26] to stable $H(q_1, q_2)$. Suppose that $H(q_1^o, q_2^o)$ is unstable. If (q_1^o, q_2^o) lies in the upper sector then $H(q_1^o - s_1, q_2^o)$ and $H(q_1^o, q_2^o + s_2)$ are also unstable, where $s_1 \in [0, q_1^o]$ and $s_2 \in [0, \infty)$ respectively. If (q_1^o, q_2^o) lies in the lower sector then $H(q_1^o + s_1, q_2^o)$ and $H(q_1^o, q_2^o - s_2)$ are also unstable, where $s_1 \in [0, \infty)$ and $s_2 \in [0, q_2^o]$ respectively. Due to the so–called "overheating" effect[29] for any given $H(q_1, q_2)$ there exist $s, s' \geq 0$ such that $H(q_1 + s, q_2)$ and $H(q_1, q_2 + s')$ would be unstable.

All properties mentioned above are established rigorously, except the fact that $\mathfrak{q}_{1,2} \neq 0$. Utilizing the analysis in Refs. 24 and 25 one can prove that $H(q_1, 1)$ is unstable if both of the following inequalities are fulfilled

$$q_1^2 < \frac{3}{16}\frac{\mu_{23}}{\mu}, \tag{77}$$

$$\frac{6\mu}{\mu_{23}}q_1\left\{1 + \frac{4q_1\sqrt{\mu}}{\sqrt{3\mu_{23}} - 4q_1\sqrt{\mu}}\right\} < 1. \tag{78}$$

Note, that from (77) it automatically follows that $(q_1, 1)$ lies in the upper sector. From (77)–(78) it follows that for q_1 small enough $H(q_1, 1)$ is unstable, thus $\mathfrak{q}_1 \neq 0$ and (77)–(78) can be used to derive the lower bound on $\mathfrak{q}_1$. Similarly, by interchanging the indices $1 \leftrightarrow 2$ in (77)–(78) one can get the lower bound on $\mathfrak{q}_2 \neq 0$.

Using the results from the previous sections we can prove (see also the discussion in Sec. 5 in Ref. 3)

Theorem 2. *Suppose $(\mathfrak{q}_1, \mathfrak{q}_2)$ lies on the stability border in the upper (resp. lower) sector. (a) If $\mathfrak{q}_2 > 1$ (resp. $\mathfrak{q}_1 > 1$) then $H(\mathfrak{q}_1, \mathfrak{q}_2)$ has a bound state at threshold. (b) If $\mathfrak{q}_1 < \mathfrak{q}_1$ (resp. $\mathfrak{q}_2 < \mathfrak{q}_2$) then $H(\mathfrak{q}_1, \mathfrak{q}_2)$ has no bound states at threshold.*

Proof. Let us prove (a). In the vicinity of $(\mathfrak{q}_1, \mathfrak{q}_2)$ one takes a sequence $(q_{1,2}(\lambda_n) \to \mathfrak{q}_{1,2})$ so that $H\big(q_1(\lambda_n), q_2(\lambda_n)\big)$ is stable. For the sequence ψ_n in R1 we take the normalized ground states of $H\big(q_1(\lambda_n), q_2(\lambda_n)\big)$. It is straightforward to check that all conditions of Theorem 1 can be satisfied. (The requirement R4 can be easily checked since the exact expressions for the ground state wave functions of the particle pairs $\{1, 3\}$ and $\{2, 3\}$ are known).
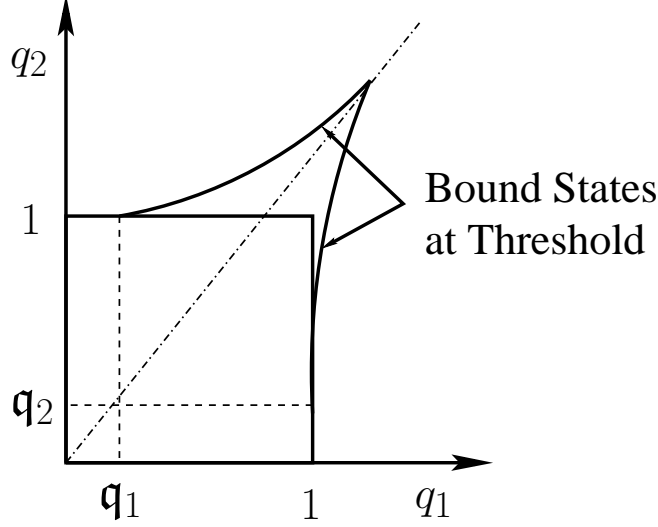
FIG. 1. The sketch of stability diagram for three Coulomb charges $\{q_1, q_2, -1\}$. Systems on the dash–dotted line have equal dissociation thresholds. The area confined by the unit square and two joint arcs represents stable systems. On the arcs of stability curve where either $q_1 > 1$ or $q_2 > 1$ there are bound states at threshold. See also the discussion in Ref. 3.

Now let us prove (b). Suppose that $(\mathbf{q}_1, 1)$ lies on the stability border in the upper sector and $\mathbf{q}_1 < \mathfrak{q}_1$. Assume by contradiction that there is a normalized $\phi \in D(H)$ such that $H(\mathbf{q}_1, 1)\phi = E_{thr}(\mathbf{q}_1, 1)\phi$ Let us rewrite (76) for $q_2 = 1$ as

$$H(q_1, 1) = H_{thr} - \frac{1}{2\mu}\Delta_R + W, \tag{79}$$

$$H_{thr} := -\frac{1}{2\mu_{23}}\Delta_\xi - \frac{1}{|\xi|}, \tag{80}$$

$$W(q_1) := -\frac{q_1}{|(1-s)\xi - R|} + \frac{q_1}{|a\xi + R|}. \tag{81}$$

In the upper sector $E_{thr}(q_1, 1) = E_0$ is constant and $H_{thr} \geq E_0$. Using that $-(\phi, \Delta_R, \phi) > 0$ we get from (79) that $(\phi, W(q_1)\phi) < 0$. Thus $(\phi, H(\mathbf{q}_1 + \varepsilon, 1)\phi) < E_{thr}$, where $\varepsilon = (\mathfrak{q}_1 - \mathbf{q}_1)/2$. Therefore, $H(\mathbf{q}_1 + \varepsilon, 1) = H(\mathfrak{q}_1 - \varepsilon, 1)$ is stable, which contradicts the properties of the stability border. $\qquad\square$

Remark. Suppose the conditions of Theorem 2 are fulfilled. From Fig. 1 one can see that it is possible to construct a sequence of points, which correspond to stable Hamiltonians and converge to $(\mathfrak{q}_1, \mathfrak{q}_2)$ (in the topology of $\mathbb{R}^2$). In the case (a) of Theorem 2 the ground states of these Hamiltonians would form a sequence that does not spread. In the case (b)

18

the ground states would form a totally spreading sequence[1]. Case (b) bears some similarity to the proof[27] of the absence of an L^2-eigenfunction at the bottom of the spectrum of the Hamiltonian of the hydrogen negative ion in the triplet S-sector.

B. Negative Atomic Ions

We consider the Hamiltonian of an atomic nucleus with charge Z and N_e electrons

$$H(Z, N_e) = H_0 - \sum_{i=1}^{N_e} \frac{Z}{|r_i|} + \sum_{1 \le i < j \le N_e} \frac{1}{|r_i - r_j|}, \tag{82}$$

$$H_0 = -\sum_{i=1}^{N_e} \Delta_i - \frac{1}{M} \sum_{1 \le i < j \le N_e} \nabla_i \cdot \nabla_j, \tag{83}$$

where the coordinate r_i points from the nucleus to the electron i. The total number of particles is $N_e + 1$ (the electrons are numbered from 1 to N_e and the nucleus is the particle number $N_e + 1$). We set $\hbar = 1$, $m_i = 1$, $m_{N_e+1} = M$. In the notations of (1)–(2) $\lambda = Z$ is the continuous parameter, $q_i(Z) = -1$ for $i = 1, \ldots, N_e$ and $q_{N_e+1} = Z$. The electrons are treated as spin $1/2$ fermions, the spin of the nucleus is set to zero. By $\mathcal{P}_{N_e}$ we shall denote the projection operator on the subspace of functions, which are antisymmetric with respect to the interchange of electrons' spin and spatial coordinates. The Hamiltonian (82) acts in $L^2(\mathbb{R}^{3N_e}; \mathbb{C}^{2^{N_e}})$ and is a self–adjoint operator with domain $D(H_0) = \mathcal{H}^2(\mathbb{R}^{3N_e}; \mathbb{C}^{2^{N_e}})$. We define

$$E(Z, N_e) := \inf \sigma\big(H(Z, N_e)\mathcal{P}_{N_e}\big). \tag{84}$$

The nuclear charge Z_{cr} is called *critical* if $E(Z_{cr}, N_e) = E(Z_{cr}, N_e - 1)$ and $E(Z, N_e) < E(Z, N_e - 1)$ for $Z > Z_{cr}$. It is known[28] that $Z_{cr} \le N_e - 1$ (due to the long–range attraction between the outer electron and remaining particles). For a rigorous proof on existence of the critical charge see Refs. 21, 22, and 29. Lieb[30] showed that $Z_{cr} \ge N_e/2$, and in Ref. 31 one finds the proof that $Z_{cr}/N_e \to 1$ if $N_e \to \infty$ (here one also assumes that the nucleus is infinitely heavy). It is generally conjectured that $Z_{cr} \in (N_e - 2, N_e - 1]$ throughout the periodic system, see, in particular, Refs. 32 and 33. Some experimental and theoretically estimated values of critical charge can be found in Ref. 34. Here we prove

Theorem 3. *Suppose that $Z_{cr} \in (N_e - 2, N_e - 1)$. Then there exists $\psi_0 \in D(H_0)$, $\|\psi_0\| = 1$, such that $H(Z_{cr}, N_e)\psi_0 = E(Z_{cr}, N_e - 1)\psi_0$ and $\mathcal{P}_{N_e}\psi_0 = \psi_0$.*

Proof. We need to show that the conditions of Theorem 1 are fulfilled. Since $Z_{cr} > N_e - 2$ by assumption, it is known[28] that there exists $\varepsilon > 0$ such that $E(Z_{cr}, N_e - i) < E(Z_{cr}, N_e - i - 1) - \varepsilon$ for $i = 1, \ldots, N_e - 1$. Due to the continuous dependence of the energies on Z there exist $z_0 > 0$ and $|\Delta\epsilon| \in (0, 2\varepsilon)$ such that for all $Z_n = Z_{cr} + z_0/n$, where $n = 1, 2, \ldots$ one has $Z_n \in (N_e - 2, N_e - 1)$ and

$$E(Z_n, N_e - i) < E(Z_n, N_e - i - 1) - 2|\Delta\epsilon| \qquad (i = 1, \ldots, N_e - 1). \tag{85}$$

The requirement R1 is fulfilled if for ψ_n we choose the normalized ground state of $H(Z_n, N_e)$ that is $H(Z_n, N_e)\psi_n = E(Z_n, N_e)\psi_n$, where, clearly, $\mathcal{P}_{N_e}\psi_n = \psi_n$. By the HVZ theorem $E_{thr}(Z) = E(Z, N_e - 1)$ for $Z = Z_n, Z_{cr}$ and ψ_n exists. R2 is obvious. The partitions $a = 1, 2, \ldots, \mathfrak{N}$, where $\mathfrak{N} = N_e$, correspond to dividing all particles into the electron number a and the rest particles. R3 follows from (85) and the HVZ theorem. Inequality (17) holds. R4 follows from Lemma 7. □

The following Lemma is essentially the result of Ahlrichs[35] generalized to the nucleus of finite mass (see also Ref. 36 for a short and clear exposition).

Lemma 7. *Suppose $H(Z, N_e)\psi = \mathcal{E}\psi$, where $\psi \in D(H_0)$, $\|\psi\| = 1$, $\mathcal{P}_{N_e}\psi = \psi$ and $\mathcal{E} < E(Z, N_e - 1) - |\Delta\epsilon|$ for some $|\Delta\epsilon| > 0$. Then $\|e^{(4CN_e)^{-1}|r|}\psi\| \leq \sqrt{2}$, where $|r| := \sum_i |r_i|$ and*

$$C := \frac{Z}{2|\Delta\epsilon|} + \frac{1}{2|\Delta\epsilon|} \left(Z^2 + 2|\Delta\epsilon|\right)^{1/2}. \tag{86}$$

Proof. Let $\mathcal{P}'_{N_e}$ denote the projection operator on the subspace of functions, which are antisymmetric with respect to the interchange of spin and spatial coordinates of the electrons $\{2, \ldots, N_e\}$. Looking at (82) it is easy to see that

$$H(Z, N_e)\mathcal{P}'_{N_e} + \frac{Z}{|r_1|}\mathcal{P}'_{N_e} \geq E(Z, N_e - 1)\mathcal{P}'_{N_e} \tag{87}$$

since in the Hamiltonian on the lhs the first electron is involved only in positive interaction terms. From (87) it follows that

$$\left(g, \left\{[H(Z, N_e) - \mathcal{E}] + Z|r_1|^{-1}\right\}g\right) \geq |\Delta\epsilon|(g, g), \tag{88}$$

where $\mathcal{P}'_{N_e}g = g$. Setting $g = f(r_1)\psi(r_1, \ldots, r_{N_e}, \sigma_1, \ldots, \sigma_{N_e})$ we get

$$\left(f\psi, [H(Z, N_e) - \mathcal{E}]f\psi\right) + Z(\psi, |f|^2|r_1|^{-1}\psi) \geq |\Delta\epsilon|(\psi, |f|^2\psi). \tag{89}$$

Using hermiticity of $i\nabla_j$ one shows[36] that

$$\left(f\psi, [H(Z, N_e) - \mathcal{E}]f\psi\right)$$

$$= \left(f\psi, [-\frac{1}{2}\Delta_1, f]\psi\right) - \frac{1}{M}\sum_{j=2}^{N_e}\left(f\psi, [\nabla_1 \cdot \nabla_j, f]\psi\right) = \frac{1}{2}\left(\psi, |\nabla_1 f|^2 \psi\right) \tag{90}$$

because $\left(f\psi, [\nabla_1 \cdot \nabla_j, f]\psi\right) = 0$ for all $j \geq 2$. Substituting (90) into (89) we produce exactly the inequality (2.6) from Ref. 36. So we can use the inequality (2.20) from Ref. 36, which in our notations reads

$$\frac{(\psi, |r_1|^{n+1}\psi)}{(\psi, |r_1|^n\psi)} \leq \frac{1}{2|\Delta\epsilon|}\left\{Z + \left[Z^2 + \frac{|\Delta\epsilon|}{2}(n+2)^2\right]^{1/2}\right\}, \tag{91}$$

This can be transformed into

$$(\psi, |r_1|^{n+1}\psi) \leq (n+1)C(\psi, |r_1|^n\psi), \tag{92}$$

where C is defined in (86). Since $\|\psi\| = 1$ (92) results in

$$(\psi, |r_i|^n\psi) \leq C^n n! \qquad (i = 1, \ldots, N_e). \tag{93}$$

Now using[35]

$$|r|^n \equiv \left(\sum_{i=1}^{N_e}|r_i|\right)^n \leq (N_e)^{n-1}\sum_{i=1}^{N_e}|r_i|^n \tag{94}$$

together with (93) we obtain

$$(\psi, |r|^n\psi) \leq (CN_e)^n n! \tag{95}$$

Using that $\|e^{\beta|r|}\psi\|^2 = \sum_n (2\beta)^n (n!)^{-1}(\psi, |r|^n\psi)$ and (95) we prove the Lemma. $\qquad\square$

A few remarks are in order. For $N_e = 2$ the statement of Theorem 3 was conjectured in Ref. 37 (see also[38]) and proved in Ref. 5. For $N_e \geq 3$ this result was conjectured in Ref. 32. The restriction $Z_{cr} \in (N_e - 2, N_e - 1)$ in the condition of Theorem 3 is imposed in order to keep the proof completely rigorous (otherwise to apply Theorem 1 one would need additional assumptions concerning the nature of dissociation thresholds). In fact, the same result must hold for $Z_{cr} < N_e - 2$.

Appendix A: Criteria for Non–Spreading Sequences

Recall[1] that the sequence of functions $f_n(x) \in L^2(\mathbb{R}^d)$ **spreads** if there is $a > 0$ such that

$$\limsup_{n\to\infty} \|\chi_{\{x||x|>R\}}f_n\| > a \qquad \text{for all } R > 0. \tag{A1}$$

(This definition can be found in the papers of Zhislin[28], who used the idea of spreading sequences in his proof of the celebrated HVZ theorem). From the definition it trivially follows: (a) if the sequence goes to zero in norm it does not spread; (b) if $|f_n(x)| \leq \sum_{k=1}^{\mathcal{N}} g_n^{(k)}(x)$, where each $g_n^{(k)}(x) \in L^2(\mathbb{R}^d)$ does not spread and $\mathcal{N}$ is finite, then f_n does not spread.

Lemma 8. *Suppose the sequence $f_n \in L^2(\mathbb{R}^d)$ is uniformly norm-bounded and $|f_n(x)| \leq |f_{n+1}(x)|$. Then f_n does not spread.*

Proof. Let us assume by contradiction that f_n spreads, so that (A1) holds. Let us fix n and choose R so that $\|\chi_{\{x||x|>R\}}f_n\|^2 < a^2/4$. Because the sequence f_n spreads we can find $n' > n$ such that $\|\chi_{\{x||x|>R\}}f_{n'}\|^2 > a^2/2$. Using that $|f_n|$ is non–decreasing we obtain

$$\|f_{n'}\|^2 = \|\chi_{\{x||x|\leq R\}}f_{n'}\|^2 + \|\chi_{\{x||x|>R\}}f_{n'}\|^2 \geq \|\chi_{\{x||x|\leq R\}}f_n\|^2 + \|\chi_{\{x||x|>R\}}f_{n'}\|^2$$

$$= \|f_n\|^2 - \|\chi_{\{x||x|>R\}}f_n\|^2 + \|\chi_{\{x||x|>R\}}f_{n'}\|^2 \geq \|f_n\|^2 + \frac{a^2}{4}. \tag{A2}$$

Thus for any f_n there exists such $f_{n'}$ with $n' > n$ such that $\|f_{n'}\|^2 \geq \|f_n\|^2 + a^2/4$. But this contradicts f_n being a norm-bounded sequence. $\qquad \square$

Here is a stronger version of Lemma 8.

Lemma 9. *Suppose the sequence $f_n \in L^2(\mathbb{R}^d)$ is uniformly norm-bounded and from any subsequence f_{n_k} one can extract a sub/subsequence $f_{n_{k_s}}$ such that $|f_{n_{k_s}}(x)| \leq |f_{n_{k_{s+1}}}(x)|$. Then f_n does not spread.*

Proof. Again, let us assume by contradiction that f_n spreads. It follows that for $k = 1, 2, \ldots$ and some $a > 0$ one can extract a subsequence f_{n_k} that satisfies $\|\chi_{\{x||x|\geq k\}}f_{n_k}\| > a$. On one hand, it is easy to see that every subsequence of f_{n_k} spreads. On the other hand, by condition of the Lemma f_{n_k} contains a subsequence, which is non–decreasing and uniformly bounded, and thus cannot spread by Lemma 8, a contradiction. $\qquad \square$

We also need the following

Lemma 10. *Suppose that $N \geq 3$ and a sequence $f_n \in L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s}) \otimes L^2(\mathbb{R}^3)$ is uniformly norm–bounded and does not spread. Suppose additionally that an operator sequence $A_n \colon L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s}) \to L^2(\mathbb{R}^{3N-6}; \mathbb{C}^{n_s})$ is such that $\sup_n \|e^{\alpha|x^\alpha|}A_n\| < K$, where $K, \alpha > 0$ are constants. Then the sequence $(A_n \otimes 1)f_n$ does not spread.*

22

Proof. The full set of relative coordinates for a given cluster partition is $x = (x^a, R_a)$ and $|x| := |x^a| + |R_a|$. For any given $\varepsilon > 0$ let us choose $R > 0$ so that the following inequalities hold

$$\sup_{x^a} \left[\chi_{\{x^a| \ |x^a| \geq R\}} e^{-\alpha|x^a|} \right] < \varepsilon/(2K \sup_n \|f_n\|), \tag{A3}$$

$$\left\| \chi_{\{R_a| \ |R_a| \geq R\}} f_n \right\| < \varepsilon/2. \tag{A4}$$

Note that

$$\chi_{\{x| \ |x| \geq 2R\}} \leq \chi_{\{x^a| \ |x^a| \geq R\}} \otimes 1 + 1 \otimes \chi_{\{R_a| \ |R_a| \geq R\}}. \tag{A5}$$

Using (A5) and (A3)–(A4) we obtain

$$\left\| \chi_{\{x| \ |x| \geq 2R\}} (A_n \otimes 1) f_n \right\| < \varepsilon \tag{A6}$$

for all n. $\qquad\square$

Obviously, Lemma 10 also holds if we replace $e^{\alpha|x^a|}$ with $(1 + |x^a|)^\alpha$, where $\alpha > 0$ is some power.

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